Jumps and Diffusions in Volatility: It Takes Two to Tango

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Thème 4 — Simulation
et optimisation
de systèmes complexes
Projet Omega

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Abstract:

We consider the following stochastic differential equation (S.D.E.) for describing financial data evolution:

$$dX_t = b(t, X_t)\, dt + \sigma(t) h(X_t)\, dW_t, \qquad X(0) = x$$

with a stochastic volatility $\sigma(t)$ (e.g. the combination of a diffusion and a jump process). We prove the existence and positivity of the solution of a Cox-Ingersoll-Ross type S.D.E. with time varying coefficients which is a special case of our model. From observation on X_t at times t_i (with non regular sampling scheme), we propose a non-parametric estimator for the volatility that is optimal in a certain way. We show its pointwise convergence and its asymptotic normality. We propose an estimator for the volatility jump times and prove a Central Limit Theorem. The application of these estimators to the BTP futures (Italian ten year bond futures) and Lira 1 month deposit Eurorates seems to confirm the adequacy of the proposed model.

AMS Classifications. 62M 05, 60G 35.

Key-words: Stochastic Volatility, Non-parametric Estimation, Estimation of the Volatility Jumps.

(Résumé : tsvp)

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La volatilité stochastique comme couple d'un processus à saut et d'un processus de diffusion

Résumé :

Motivés par des considérations économiques, nous proposons de modéliser l'évolution des données financières par une équation différentielle stochastique (EDS)

$$dX_t = b(t, X_t)dt + \sigma(t)h(X_t)dW_t, \qquad X(0) = x.$$

où le coefficient de diffusion (appelé volatilité) $\sigma(t)$ est stochastique (couple de diffusion continue et processus à sauts, par exemple).

Nous donnons d'abord un résultat d'existence et de positivité de la solution de l'EDS de Cox-Ingersoll-Ross à coefficients variables stochastiques. Puis, à partir de l'observation d'une trajectoire X_t à des instants discrets t_i (irrégulièrement espacés) nous proposons un estimateur non-paramétrique de la volatilité (d'un certain point de vue optimal). Nous montrons sa convergence ponctuelle et sa normalité asymptotique.

Ensuite, nous proposons un estimateur consistant des instants de saut de la volatilité pour lequel, dans le cas d'instants de sauts déterministes, nous montrons un Théorème Central Limite. Enfin, nous appliquons ces estimateurs à des données financières italiennes qui semble confirmer l'adéquation du modèle proposé.

AMS Classifications : 62M 05, 60G 35.

Mots-clé : Volatilité stochastique, Estimation non-paramétrique, Détection des sauts de la volatilité.

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1 Introduction

In recent years accurate analysis of financial time series has brought to the attention of researchers the existence of several anomalies. In particular, volatility estimation has always occupied a central place in the research on financial markets: this is due to the practical implications that an accurate measure of this quantity would have in terms of derivatives pricing. All the original models developed for pricing options and other interest rates derivatives such as the Black-Scholes model shared the feature of a constant volatility parameter. It is true that, for example, in the Cox-Ingersoll-Ross [8] term structure model, the volatility parameter is specified so that the stochastic process $\{X_t\}$ describing the underlying asset shows a state dependent evolution, i.e., higher (lower) volatility for higher (lower) level of the state process. At the same time the specification chosen in [8] keeps σ constant:

$$dX_t = \beta(\theta - X_t)dt + \sigma\sqrt{X_t}\,dW(t), \qquad X(0) = x.$$

This situation has been called into question in recent years due to two innovations which took place respectively in applied work and in theoretical finance.

On the first account we experienced the explosion of the econometric literature on ARCH-GARCH models which showed the pervasiveness of heteroskedasticity in the evolution of financial time series. On the other side, starting with Hull and White [22], a new family of models incorporates the idea that volatility follows itself a diffusion process. Bensoussan, Crouhy and Galay [2], for example, derived an extension of the Black and Scholes model where stochastic volatility arises from the impact of a change in the value of the firm's asset on the financial leverage. These developments produced a new vein of empirical research aimed at the estimation of stochastic volatility using different approaches as in Gourieroux, Monfort and Renault [18], Harvey, Ruiz and Shephard [21], and Nelson [31]. At the same time growing evidence has been collected pointing to the presence of parameters instability in the estimation of Cox, Ingersoll and Ross type of models. Brown-Dybwig [5] and Barone, Cuoco and Zautzik [1] for example find in different situations that the constancy of the long term mean of the short term interest rate (θ) and of the recal parameter (β) is not supported by the data; Fournié and Talay [14] found non constancy of the diffusion parameter [1].

More problems pointing to the same issue of parameters instability have been evidenced more recently even inside the ARCH-GARCH approach by Hamilton and Susmel [20]. What has been questioned is the autoregressive nature of the process the variance should follow. The rational motivating these criticisms is that the excessive persistence ARCH-GARCH models imply for the variance reduces their forecasting performance. In particular it has been observed that big shocks have different implications for the future evolution of the variance with respect

[1]For example the September 1995 issue of *The Journal of Fixed Income* is almost entirely devoted to the problem of parameter instability in term structure models

to small shocks. In the ARCH-GARCH set-up what we are supposed to observe is that small shocks are followed by small shocks and big shocks are followed by big shocks giving rise to the now well known volatility clustering effect first evidenced by Mandelbrot [29]. What is troublesome with this approach is that it implies higher persistence for big shocks than for small shocks due to the autoregressive nature of the volatility process. Therefore a big shock implies a longer transition period before the process returns to its *natural* state. Financial time series instead seem to suggest a much faster adjustment path. Therefore the crucial question is: *for how long does volatility persists*?

In order to give a solution to these problems Hamilton and Susmel [20] propose that the parameters of the ARCH process characterizing conditional volatility evolution may come from one of several different volatility regimes and that the transition from one regime to the other be governed by an unobserved Markov chain. In this way sudden jumps in volatility can be accomodated by the switch of the volatility process to a new regime without being forced to assume long and unnecessary transition dynamics. Even this approach, unfortunately, has its own drawbacks. In fact there is no way to identify rigorously the number of states the process under investigation may have been through. The computational requirements to perform the analysis when the number of states is greater than 4 are prohibitive.

From the analysis so far performed it appears, that more work needs to be done in order to identify the nature of volatility evolution in financial time series data. This seems particularly needed since it has been only in recent times that data on financial transactions at a very high frequency became available. This type of data are particularly apt for testing the continuous time models that characterize large part of theoretical finance. In fact so far most of the estimations of financial models were performed using daily data, which are a poor approximation for continuous time. It is our claim, supported also by Müller, Dacorogna, Davé, Olsen, Pictet, Weizsäcker [30] and Goodhart, Ito, and Payne [17], that the observation of almost continuous time real financial processes will highlight a complete different set of dynamics from the ones discrete time econometrics models brought to our attention.

The difficulties of this approach are just begining to unravel and they point out the necessity of linking continuous time stochastic process analysis with estimation. In order to reach this target we need to state our problem in terms of stochastic differential equations. We consider a stochastic process statisfying the following stochastic differential equation:

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \qquad X(0) = x. \tag{1}$$

We observe a sample path of the process $\{X_t\}$ at the discrete times t_i and we want to estimate the diffusion coefficient $\sigma(t, X_t)$.

The parametric case $\sigma(t, x) = \sigma f(x)$ is studied in Dohnal [11] for regular sampling times and by Genon-Catalot and Jacod [16] in a general framework. The non-parametric case for a non time varying diffusion coefficient, i.e., $\sigma(t, x) = \sigma(x)$, has been treated by Florens-Zmirou

[13] for one dimensional process and by Brugière [6] for multidimensional processes. Fournié and Talay [14] tried to estimate the Cox, Ingersoll and Ross model with constant coefficients on real financial data (daily observations of the French short term interest rates) and they concluded that, at least the diffusion coefficient was time varying.

Florens-Zmirou [12] gave a non-parametric estimator for a time varying volatility $\sigma(x,t) = \sigma(t)h(x)$. Genon-Catalot, Laredo and Picard [15] introduced the non-parametric estimation by wavelet method. In a companion paper Bertrand [3] compares the estimator proposed here with a wavelet estimator in terms of their Mean Integral Square Errors (MISE) and proves that the one proposed here is better as soon there is at least a volatility jump and it is more robust.

Both Florens-Zmirou [12] and Genon-Catalot, *et al.* [15] consider a time varying coefficient $\sigma(t)$ which is a $\mathcal{C}^1$ function of time and deterministic. However, for financial applications it seems more reasonable to consider a stochastic volatility $\sigma(t)$. In this case the coefficent $\sigma(t)$ would be less regular than $\mathcal{C}^1$. We will consider both a Hölder continuous function of time and a piecewise constant function corresponding either to a continuous diffusion either to a jump process. A description of stochastic volatility which is the same as the one used in this paper is considered by El Karoui and Jeanblanc-Picqué [9]. As an example we can think of a Wiener process (W_t) with the standard filtration $(\mathcal{G}_t^W)$ and $\sigma(t)$ another stochastic process independent of the increments of W_t. Let $(\mathcal{F}_t)$ be the enlarged filtration, $\sigma(t)$ is $\mathcal{F}_t$-adapted and (W_t) is still a Wiener process for this filtration.

The plan of the paper is the following. In Section 2 we describe the model we will use and we give some results on the existence and positiveness of the time varying coefficients stochastic differential equation which characterizes the Cox, Ingersoll and Ross model. In Section 3 we describe the kernel estimator we use for irregular sampling times, we prove its point-wise convergence and study its Mean Integral Square Error. In Section 4 we make some numerical simulations of the volatility estimator. In Section 5 we study the estimation problem for jump times. Finally in Section 6 we apply our method to real financial data[2].

Most proofs are given in the appendixes. In Appendix A, we prove the existence and positiveness of the time varying coefficients Cox, Ingersoll and Ross stochastic differential equation (Theorem 2.1). In Appendix B we give some bounds which are used in Appendix C to prove Central Limit Theorem for point-wise convergence of the volatility estimator (Theorem 3.1). In Appendix D, we prove a Central Limit Theorem for the volatility jump time estimator (Theorem 5.1) when the jump times are deterministic (but with random jumps).

[2]In this paper we use *off-line* estimation in every case.

2 Description of the Model

We assume that there exists a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a one dimensional Wiener process (W_t) adapted to $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We will denote by Leb the Lebesgue measure on $\mathbb{R}$. We consider a stochastic process statisfying the following stochastic differential equation:

$$dX_t = b(t, X_t)\,dt + \sigma(t)h(X_t)\,dW_t \tag{2}$$

where the function $h(\cdot)$ is assumed to be known, the volatility coefficient $\sigma(\cdot)$ is an unknown function of time and has to be correctly estimated, the drift coefficient $b(t, x)$ could be unknown.

We observe one sample path of the process $(X_t, t \in [0, T])$ at irregular sampling times t_i for $i = 0, .., N$. We denote $\Delta_i = t_{i+1} - t_i$ and $\|\Delta\| = \sup_{i \in [0,N]} \Delta_i$. We assume that $\|\Delta\|$ is small in comparison to T.

We impose the following assumptions:

(A0) $\sigma(t)$ *is adapted to the filtration* $\mathcal{F}_t$, $b(t, .)$ *is a non-anticipative map,* $b \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$ *and* $\exists L_T > 0$ *such that* $\forall t \in [0, T], \quad \mathbb{E}\sigma^4(t) \leq L_T$.

Moreover, we want to consider both a jump process and a diffusion process, respectively corresponding to the following assumptions:

(A1) $\sigma(t) = \sum_{\rho=0}^{f} \sigma_\rho \, \mathbf{1}_{[t_\rho, t_{\rho+1})}(t)$ *where* t_ρ *are the jump times. We assume there is a finite number of jumps, i.e.* $f < \infty$. *We denote the jump* $\delta\sigma^2_\rho = \sigma^2_{\rho-1} - \sigma^2_\rho$.

If we assume that the volatility jump times correspond to the sampling times t_i, we have:

(A1') $\sigma(t) = \sum\limits_{i=0}^{N} \sigma_i \, \mathbf{1}_{[t_i, t_{i+1})}(t)$.

(A2) $\exists m > 0$ *such that* $\sigma^2(\cdot)$ *is almost surely Hölder continuous of order* m *with respect to a constant* $K(\omega)$ *and such that* $\mathbb{E}K(\omega)^2 < \infty$.

Remark : If $\sigma(t)$ (or $\sigma(t)^2$) satisfies a stochastic differential equation then (A2) is fulfilled, see for example Revuz and Yor [32, th. 2.1, p. 25].

We need to check the existence of $\int\limits_{t_i}^{t_{i+1}} b^4(s, X_s)ds$, so we will impose the following conditions:

(B1) $$\exists K_T > 0, \quad \forall t \in [0, T], \quad \mathbb{E}|b(t, X_t)|^4 \leq K_T.$$

In Section 5, we will use assumption *(A3)* defined for convenience on page 16. In the sequel we work on the simplified model:

$$dX_t = b_1(t, X_t)dt + \sigma(t)\,dW_t, \qquad X(0) = x. \tag{3}$$

In fact, under reasonable assumptions, the model (2) becomes (3) after a change of variable.

Proposition 2.1 *Assume that there exists a domain $D \subseteq I\!R$ such that $h \in \mathcal{C}^1(D, I\!R^+)$, $h^{-1} \in L^1_{loc}(D)$ and verifying $I\!P(X_t, \in D, \ \forall t \in [0,T]) = 1$, where X_t is the solution of (2). Let $H(x) = \int h^{-1}(\xi)d\xi$, then $Y_t = H(X_t)$ satisfies (3), with $b_1(t,x) = h^{-1}(x)a(t,x) - \frac{1}{2}h'(x)\sigma^2(t)$.*

Proof : It follows directly from Itô's Formula. ■

Applications

Example 1: Let S_t be an asset price satisfying the following stochastic differential equation

$$dS_t = S_t\left[a(t,S_t)dt + \sigma(t)dW_t\right], \qquad X(0) = S_0.$$

Then $X_t = \ln(S_t)$ satisfies (3) with $b(t,x) = a(t, e^{(x)}) - \frac{1}{2}\sigma^2(t)$ and $b(t,x)$ fulfills the assumption (B1).

Example 2: Let X_t be an asset price satisfying the stochastic differential equation

$$dX_t = \left[c(t)\alpha(t) - X_t\right]\, dt + \sigma(t)\sqrt{X_t}\, dW_t, \qquad X(0) = X_0. \tag{4}$$

i.e., the Cox, Ingersoll and Ross differential equation with time varying parameters where $c(), \alpha(t)$, and $\sigma(t)$ are $\mathcal{F}_t$ adapted. If $I\!P(X_t > 0, \forall t \in [0,T]) = 1$, then $Y_t = 2(X_t)^{1/2}$ satisfies (3) with $b(t,x) = 2x^{-1}\left[c(t)\alpha(t) - \frac{1}{4}\sigma^2(t)\right] - \frac{1}{2}\alpha(t)x$.

We deduce the positiveness of X_t in the case stochastic $c(t), \alpha(t)$ and $\sigma(t)$ from the following generalisation of the well-known result for the time constant case.

THEOREM 2.1 *Assume that $c(t)$, $\sigma(t)$ and $\alpha(t)$ are $\mathcal{F}_t$-adapted, there exist $K, \lambda > 0$, such that $\forall t > 0$, $\mid c(t) \mid \leq K$ and $\lambda \leq \sigma(t) \leq K$, $\forall t > 0$, $\sigma(t)^2 \leq 2c(t)\alpha(t)$ and $X(0) > 0$ a.s.. Then (4) has a unique strong solution and $I\!P(X_t > 0, \forall t > 0) = 1$.*

Proof: see Appendix A. ■

We need some stronger assumptions to obtain (B1).

3 Volatility Estimation

3.1 Description of the Estimator

We give a description of the estimator depending explicitely on the the number of observations taken into account to estimate $\sigma(t)$. The number of observations taken into account is defined as the window size and denoted by A.

The non-parametric estimator for the volatility is defined as follow:

$$S_{A,\Delta}(t) = \sum_{j=A/2}^{N-A/2} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \frac{(X_{t_{j+i+1}} - X_{t_{j+i}})^2}{\Delta_{j+i}} \right\} \mathbf{1}_{[t_j,t_j+1)}(t) \tag{5}$$

It is a symmetric kernel type estimator which generalizes the estimator for regular sampling interval given in [3]. In the following we extend the results given in [3] to the case of non constant sampling intervals.

Remark 3.1:

i) The estimator $S_{A,\Delta}(t_j)$ is defined by (5) for $t \in [t_{j-A/2}, t_{j+A/2}]$.

ii) At each time t_j, $S_{A,\Delta}(t_j)$ is the average of the last $A/2$ and of the next $A/2$ instantaneous quadratic variations $\Delta_{j+i}^{-1}(X_{t_{j+i+1}} - X_{t_{j+i}})^2$; therefore the estimator $S_{A,\Delta}(t_j)$ is $\mathcal{F}_{t_{j+A/2}}$ adapted.

3.2 Construction of the Centred Kernel Estimator

In the case of regular sampling, $t_i = i/n$, the kernel estimator for (3) given in [12] is :

$$S_n(t,K) = n \left\{ \sum_{i=1}^{n} K\left[\frac{\frac{i}{n} - t}{h_n}\right] \right\}^{-1} \sum_{i=1}^{n} K\left[\frac{\frac{i}{n} - t}{h_n}\right] (X_{\frac{i+1}{n}} - X_{\frac{i}{n}})^2 \tag{6}$$

where K is a compactly supported kernel. Florens-Zmirou [12, Th. 3, p. 200] shows that for a deterministic $\mathcal{C}^1$ function $\sigma(\cdot)$,

$$(nh_n)^{1/2} S_n(t) \Rightarrow \mathcal{N}(0, \|K\|_{L^2}^2\, \sigma^2(t))$$

Let $\mathcal{L}$ = Leb (supp K), we remark that the flat kernel $K_0 = \mathcal{L}^{-1} 1_{[0,1]}$ is an optimal kernel. Indeed, $S_n(\cdot, \lambda K) = S_n(., K)$, thus we can impose the condition:

$$1 = \int K(x)dx \leq \|K\|_{L^2(\text{supp }\ K)} \|\mathbf{1}\|_{L^2(\text{supp }\ K)}$$

Therefore, $\|K\|_{L^2} \geq \mathcal{L}^{-1/2} = \|K_0\|_{L^2}$. If we choose $\mathcal{L} = 1$, $K = \mathbf{1}_{[-1/2,1/2]}$, denote $\Delta = 1/n$ and $A = [nh_n]$, we obtain (5).

3.3 Local Consistency of Estimator

At each point of continuity of σ, we have the following convergence result:

<u>**THEOREM**</u> **3.1** *Assume that (A0), (B1) are satisfied,* $A\|\Delta\| \to 0$ *and* $A \to \infty$. *Then* $S_{A,\Delta}(t) \to 1/2\,[\sigma^2(t_+) + \sigma^2(t_-)]$ *almost surely, where* $\sigma^2(t_-)$ *(respectively* $\sigma^2(t_+)$*) denotes the left (respectively right) limit. Moreover*

i) if (A1) holds and $\mathbb{P}(t_\rho = t) = 0$, then

$$A^{1/2}\frac{[S_{A,\Delta}(t) - \sigma^2(t)]}{\sigma^2(t)} \Rightarrow \mathcal{N}(0, \sqrt{2}) \tag{7}$$

ii) If (A2) holds and $\Delta A^{1+1/2m} \to 0$, then (7) holds.

This generalizes the result on point-wise convergence from [12], [7] to the stochastic volatility case.

Proof: In order to prove the theorem we decompose the estimator into three terms. (See Proposition 3.1 below). The three terms are separately studied in Lemma C.1, C.2, and C.3 in Appendix so that the theorem follows from the proposition and Lemmas. ■

Proposition 3.1 *Assume that (A0) is satisfied. Then we have*

$$S_{A,\Delta}(t) = M_{A,\Delta}(t) + N_{A,\Delta}(t) + D_{A,\Delta}(t) \tag{8}$$

where:

$$M_{A,\Delta}(t) = \sum_{j=A/2}^{N-A/2} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \overline{\sigma}^2_{j+i} \right\} \mathbf{1}_{[t_j,t_{j+1})}(t) \tag{9}$$

$$N_{A,\Delta}(t) = 2 \sum_{j=A/2}^{N-A/2} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \xi_{j+i} \right\} \mathbf{1}_{[t_j,t_{j+1})}(t) \tag{10}$$

$$D_{A,\Delta}(t) = 2 \sum_{j=A/2}^{N-A/2} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \eta_{j+i} \right\} \mathbf{1}_{[t_j,t_{j+1})}(t) \tag{11}$$

and

$$\overline{\sigma}^2_i = \Delta_i^{-1} \int_{t_i}^{t_{i+1}} \sigma^2(s)\, ds \tag{12}$$

$$\xi_i = \Delta_i^{-1} \int_{t_i}^{t_{i+1}} \sigma(s) \left[\int_{t_i}^{s} \sigma(u)\, dW_u \right] dW_s \tag{13}$$

$$\eta_i = \Delta_i^{-1} \left\{ \int_{t_i}^{t_{i+1}} b(s, X_s)(X_s - X_{t_i})ds + \int_{t_i}^{t_{i+1}} \sigma(s) \left[\int_{t_i}^{s} b(u, X_u)\, du \right] dW_s \right\} \tag{14}$$

Proof: Applying Itô's formula on $(X_s - X_{t_i})^2$, we get (8). ■

Remark : The function $M_{A,\Delta}(t)$ only depends on $\overline{\sigma}^2_i$, the average value of $\sigma^2(t)$ on the interval $[t_i, t_{i+1}[$. The properties of the random variables (ξ_i) and (η_i) are given in Appendix B.

3.4 Mean Integral Square Error

To analyze more precisely the rate of convergence of the estimator, we look at the Integral Square Error (ISE) and its mean value (MISE) :

Definition 1 *For a given weight function $\gamma(\cdot) \geq 0$, we define :*

$$\mathrm{ISE}(\gamma, A, \Delta) = \int_0^T \gamma(t) \left[S_{A,\Delta}(t) - \sigma^2(t)\right]^2 dt$$

$$\mathrm{MISE}(\gamma, A, \Delta) = \mathbb{E} \int_0^T \gamma(t) \left[S_{A,\Delta}(t) - \sigma^2(t)\right]^2 dt$$

Let us define some notations. Without any loss of generality we assume that $\gamma(\cdot)$ is piecewise constant on $[t_j, t_{j+1})$ with value 0 or 1. We define:

$$R_1(\gamma, A, \Delta) = \int_0^T \gamma(t) \left[M_{A,\Delta}(t) - \sigma^2(t)\right]^2 \, dt \tag{15}$$

$$R_2(\gamma, A, \Delta) = \int_0^T \gamma(t) \, N^2_{A,\Delta}(t) \, dt \tag{16}$$

$$R_3(\gamma, A, \Delta) = \int_0^T \gamma(t) \, D^2_{A,\Delta}(t) \, dt \tag{17}$$

In the deterministic case without drift, $\mathbb{E} R_1(\gamma, A, \Delta)$ corresponds to the bias term and $\mathbb{E}\left[R_2(\gamma, A, \Delta) + R_3(\gamma,\right.$ to the variance term.

Since $D_{A,\Delta}(t)$ contains all the terms depending on $b(t, x)$, if (B1) holds, the drift term is negligeable in the Integral Square Error, as stated below.

Proposition 3.2 *Assume that (A0) and (B1) are satisfied. Then:*

$$\mathrm{ISE}(\gamma, A, \Delta) = R_1(\gamma, A, \Delta) + R_2(\gamma, A, \Delta) + \epsilon$$

where

$$\mathbb{E}|\epsilon| \leq C \|\Delta\| \left\{\mathbb{E} R_1(\gamma, A, \Delta) + \mathbb{E} R_2(\gamma, A, \Delta)\right\}^{1/2}$$

Proof : This result follows from the Hölder inequality, Lemma C.2 and from

$$\int_0^T \gamma(t) N_{A,\Delta}(t) \left[\sigma^2(t) - M_{A,\Delta}(t)\right] dt = 0$$

■

Since C does not depend on A, optimizing the Integral Square Error is equivalent to optimizing $\mathbb{E} R_1(\gamma, A, \Delta) + \mathbb{E} R_2(\gamma, A, \Delta)$. We turn now to those two terms.

Proposition 3.3 *Assume that (A0) is satisfied. Then*

$$\mathbb{E}R_1(\gamma, A, \Delta) + \mathbb{E}R_2(\gamma, A, \Delta) = \tag{18}$$

$$\sum_{j=0}^{N-1} \gamma(t_j)\Delta_j V_j^2 + \sum_{j=0}^{N-1} \gamma(t_j)\Delta_j \mathbb{E}\left[\overline{\sigma}_j^2 - A^{-1} \sum_{i=-A/2}^{A/2} \overline{\sigma}_{j-i}^2\right]^2 + 4A^{-1} \sum_{j=0}^{N-1} \alpha_j \, \Delta_j (\mathbb{E}\xi_j^2)$$

where

$$V_j^2 = \Delta_j^{-1} \int_{t_j}^{t_{j+1}} \left[\sigma^2(s) - \overline{\sigma}_j^2\right]^2 ds$$

and

$$\alpha_j = A^{-1} \sum_{i=-A/2}^{A/2} \gamma(t_{j+i})\Delta_{j+i}$$

If moreover (A2) holds, then $R_1(\gamma, A, \Delta) \leq (A\|\Delta\|)^{2m} \|\gamma\|_{L^1} K^2(\omega)$.

Proof : From (10) and (27) (see Lemma B.2 in Appendix) , we have

$$\begin{aligned}
\mathbb{E}R_2(\gamma, A, \Delta) &= 4 \sum_{j=0}^{N-1} \gamma(t_j) \, \Delta_j \, \mathbb{E}(A^{-1} \sum_{i=-A/2}^{A/2} \xi_{j+i})^2 \\
&= 4A^{-2} \sum_{j=0}^{N-1} \gamma(t_j) \, \Delta_j \sum_{i=-A/2}^{A/2} \mathbb{E}(\xi_{j+i})^2 \\
&= 4A^{-1} \sum_{j=0}^{N-1} \alpha_j \, \mathbb{E}(\xi_j)^2
\end{aligned}$$

We turn now to the term $R_1(\gamma, A, \Delta)$. Let $\mu_j = \sum_{i=-A/2}^{A/2} \overline{\sigma}_{j+i}^2$, we have:

$$R_1(\gamma, A, \Delta) = \sum_{j=0}^{N-1} \gamma(t_j)\Delta_j \int_{t_j}^{t_{j+1}} \left[\mu_j - \sigma^2(s)\right]^2 ds$$

combined with

$$\Delta_j^{-1} \int_{t_j}^{t_{j+1}} \left[\mu_j - \sigma^2(s)\right]^2 ds = V_j^2 + \left[\mu_j - \overline{\sigma}_j^2\right]^2$$

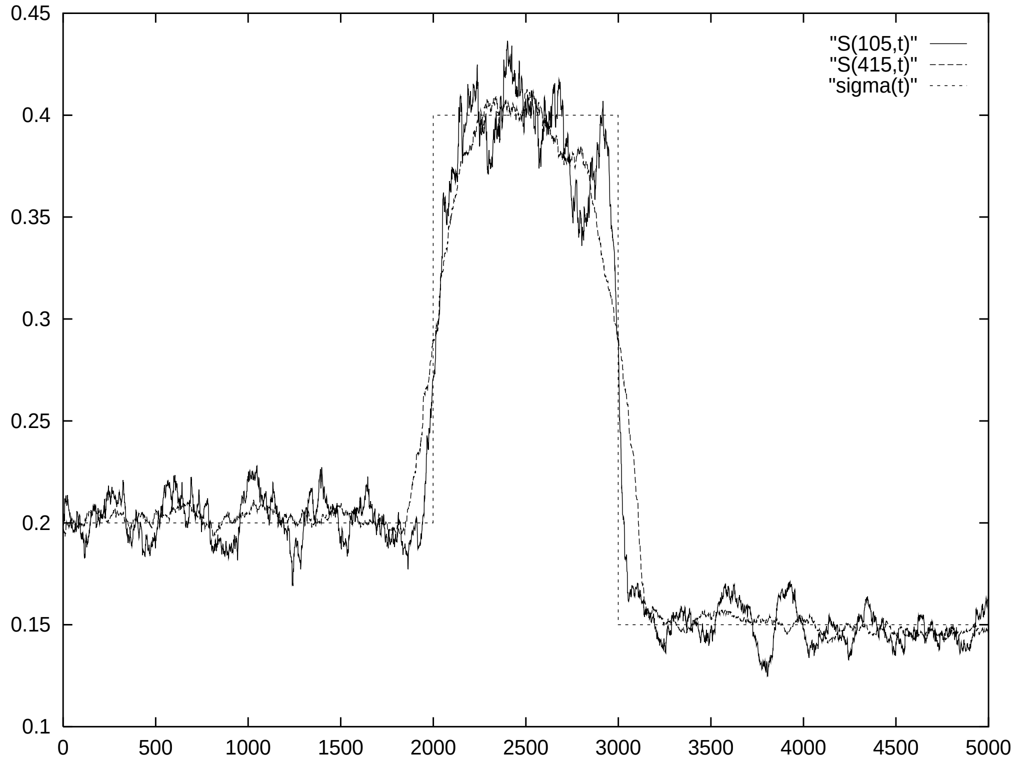
we deduce (18). ∎

Remark : Replacing $\sigma^2(t)$ by its average values $\overline{\sigma}_j^2$ induces on $R_1(\gamma, A, \Delta)$ an error of $\sum_{j=0}^{N-1} \gamma(t_j)\Delta_j V_j^2$, which is bounded either by $C\|\Delta\|$ when (A1) is fulfilled either by $K(\omega)\|\Delta\|^{2m}$ when (A2) holds. This means that when $\sigma(t)$ is a diffusion process observed at discrete times, we do not never see the local structure of $\sigma(t)$ but we are only able to estimate the average of $\sigma(t)$ between two successive observation times t_i and t_{i+1}.

4 Some Numerical Simulations

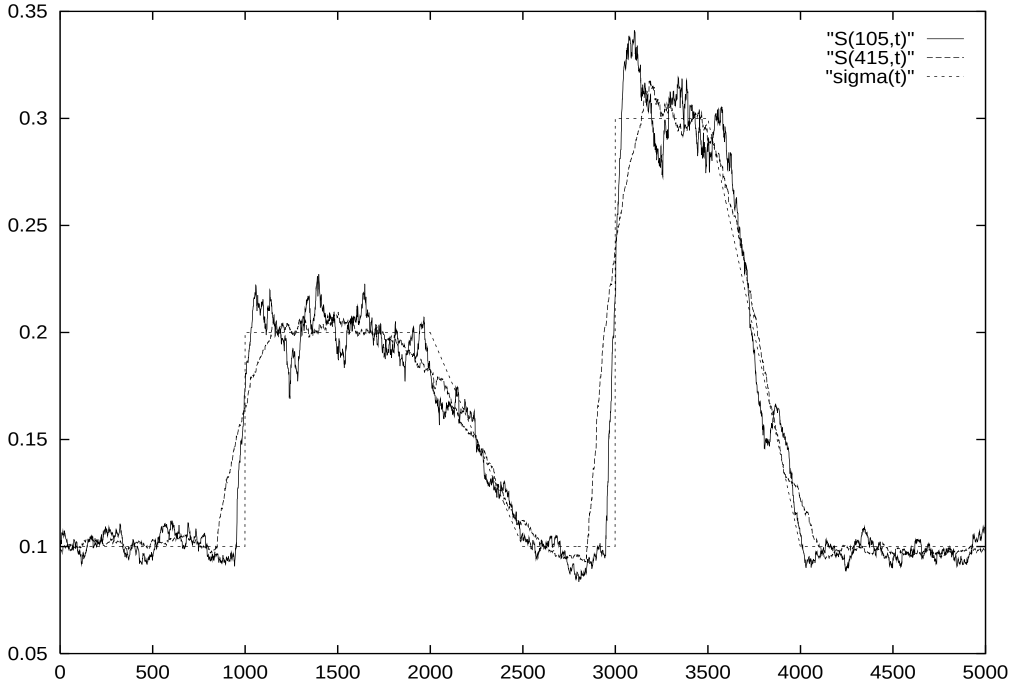
In this section, we consider a regular sampling scheme. We assume that $T = 1$ and fix $N = 5000$, $\Delta = 1/N$. We simulate numerically a path of the process X_t satisfying (4) using the Euler method with step $\Delta/10$. We consider two different specifications of volatility evolution. For each one we plot two estimators corresponding to two different window sizes, i.e., $A = 105$ and $A' = 3A$. The first one is an example of a piecewise constant volatility.

Example 4.1: Piecewise Constant Volatility with Jumps



The second example is the case of jumps in volatility followed by a continuosly decreasing volatility. We take this as an example of jumps and diffusions in volatility. Notice that, after the second jump, volatility decrease particularly fast so that, *a priori*, it seems difficult to distinguish this from a case of a jump. However our estimator seems to be able to distinguish this from a volatility jump.

Example 4.2: Volatility Jumps and Diffusions



5 Volatility Jump Times Estimation

For the case showed in example 4.1, i.e., volatility jump, the simulations just presented suggest us that by varying the size of the windows we obtain two non-parametric estimations of volatility which cross each other in a neighborhood of the volatility jump time. Now we will make this heuristic remark more precise.

Let λ be a fixed integer $\lambda \geq 2$, from Proposition 3.1 we have:

$$\begin{aligned} S_{A,\Delta}(t_j) - S_{\lambda A,\Delta}(t_j) &= M_{A,\Delta}(t_j) - M_{\lambda A,\Delta}(t_j) \\ &+ N_{A,\Delta}(t_j) - N_{\lambda A,\Delta}(t_j) + D_{A,\Delta}(t_j) - D_{\lambda A,\Delta}(t_j) \end{aligned} \tag{19}$$

From Lemma C.2, and C.3 we have

$$I\!\!E|N_{A,\Delta}(t_j) - N_{\lambda A,\Delta}(t_j)|^2 \leq KA^{-1}$$

and

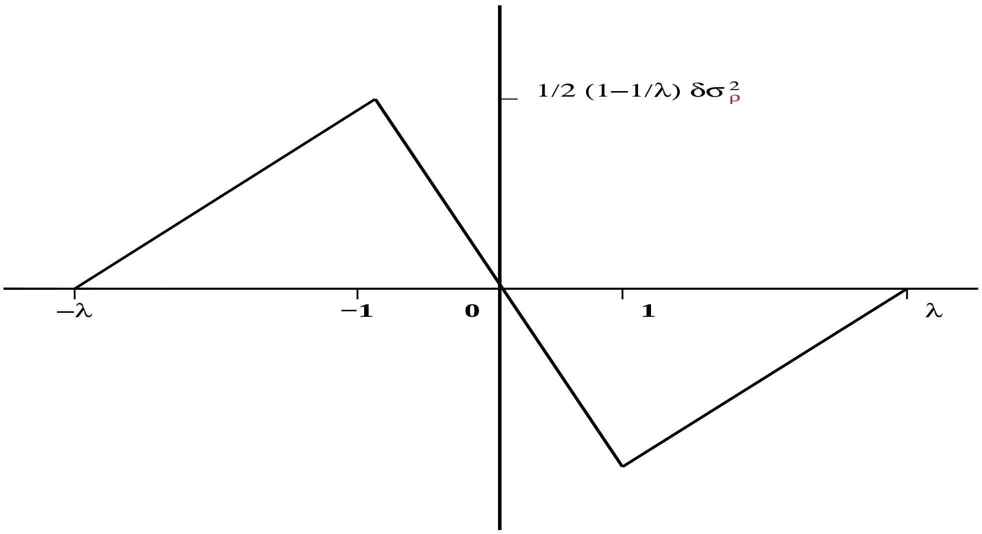
$$I\!\!E|D_{A,\Delta}(t_j) - D_{\lambda A,\Delta}(t_j)|^2 \leq K\|\Delta\|$$

Therefore these two terms could be disregarded when $A \to \infty$ and $\|\Delta\| \to 0$. At this point we are left with the first term which is not small. Let t_ρ be the volatility jump time and assume it is an isolated jump time i.e., there is no other jump time in the interval $[t_{\rho-\lambda A}, t_{\rho+\lambda A}]$. We have:

$$M_{A,\Delta}(t_j) - M_{\lambda A,\Delta}(t_j) = \phi(A^{-1}(j-\rho)) \tag{20}$$

where $\phi(\cdot)$ is an affine function vanishing at zero defined in the following figure, when $\delta\sigma_\rho^2 > 0$. Elsewhere we should replace $\phi(.)$ by $-\phi(.)$ in (20).

Representation of the function $\phi(t)$



This explains why the two estimators cross each other in a neighbourhood of t_ρ. For this reason, we define our **volatility jump time estimator** as a crossing time of the centred kernel estimator with two different window sizes, A and $\lambda\, A$, where λ is a fixed integer $\lambda \geq 2$.

More precisely we define

$$\begin{aligned} c(A,\Delta) \quad \in \quad & \{i \text{ s. t. } [S_{\lambda A,\Delta}(t_i) - S_{A,\Delta}(t_i)] \times [S_{\lambda A,\Delta}(t_{i+1}) - S_{A,\Delta}(t_{i+1})] \leq 0 \\ & \text{and } \mid S_{\lambda A,\Delta}(t_{i+1}) - S_{A,\Delta}(t_{i+1}) \mid > 0\} \end{aligned}$$

The index $c(A,\Delta)$ correspond to the crossing time denoted by $t_c(A,\Delta)$. Let $B_{A,\Delta} = \{\omega$ such that $\exists c$ with $S_{A,\Delta}($ $S_{\lambda A,\Delta}(t_c)$ and $|c-\rho| \leq A/2\}$. Moreover we introduce the following assumption:

(A3) (A1') is fulfilled and t_ρ is the only jump time in $[t_{\rho-\lambda A}, t_{\rho+\lambda A}]$.

Proposition 5.1 *Assume that (A0), (A1'), (B1) and (A3) are satisfied. Then there exists $K_\lambda > 0$ such that*

$$I\!\!P(B_{A,\Delta} \cap \{|\delta\sigma_\rho^2| \geq \kappa\}) \geq 1 - K_\lambda\, \kappa^{-2}\, (A^{-1} + \|\Delta\|)$$

Proof : Let us define

$$g_A(j) \quad = \quad S_{A,\Delta}(t_j) - S_{\lambda A,\Delta}(t_j) \tag{21}$$

$$L_A(j) \quad = \quad A^{1/2}\, [N_{A,\Delta}(t_j) - N_{\lambda A,\Delta}(t_j)] \tag{22}$$

$$\mathcal{D}_A(j) \quad = \quad D_{A,\Delta}(t_j) - D_{\lambda A,\Delta}(t_j) \tag{23}$$

To fix the idea, let $\delta\sigma_\rho^2 > 0$, if $g_A(\rho - A/2) > 0$ and $g_A(\rho + A/2) < 0$ then $\omega \in B_{A,\Delta}$. Therefore

$$\mathbb{P}(B^c_{A,\Delta}) \leq \mathbb{P}(g_A(\rho - A/2) < 0) + \mathbb{P}(g_A(\rho + A/2) > 0)$$

The two probabilities can be bounded in the same way. We just consider the first one. From assumption (A3) and (20), we have:

$$g_A(\rho - A/2) = \frac{1}{2}(1 - \lambda^{-1})(\delta\sigma_\rho^2) + A^{-1/2} L_A(\rho - A/2) + \mathcal{D}_A(\rho - A/2)$$

This implies

$$\begin{aligned}
&\mathbb{P}(g_A(\rho - A/2) < 0) \\
&\qquad \leq \ \mathbb{P}(|A^{-1/2} L_A(\rho - A/2) + \mathcal{D}_A(\rho - A/2)| > \tfrac{1}{2}(1 - \lambda^{-1})|\delta\sigma_\rho^2|) \\
&\qquad \leq \ 4(1 - \lambda^{-1})\kappa^{-2}\, \mathbb{E}[|A^{-1/2} L_A(\rho - A/2) + \mathcal{D}_A(\rho - A/2)|^2] \\
&\qquad \leq \ K_\lambda\, \kappa^{-2}\, (A^{-1} + \|\Delta\|).
\end{aligned}$$

■

The above proposition induces the consistency of our jump time estimator, as stated below.

Corollary 5.2 *Assume that (A0), (A1') and (B1) are satisfied. If* $A \to \infty$ *and* $A\|\Delta\| \to 0$ *then* $\lim_{A\to\infty,\, A\|\Delta\|\to 0} t_c(A, \Delta) = t_\rho$ *in Probability.*

Proof: If (A1') is satisfied for a given sampling procedure, say, Δ_0, then it is also fulfilled for every refined sampling procedure Δ. So the asymptotic $\|\Delta\| \to 0$ is meaningful and hereafter $A \to \infty$ and $A\|\Delta\| \to 0$.

Since there is only a finite number of jumps, for each fixed $\lambda \in \mathbb{N}^*$, for $A\|\Delta\|$ small enough, the assumption (A3) holds. Indeed, let $\delta_0 := \inf_{t_i \in \Delta_0} \mid t_{i+1} - t_i \mid$, (A3) is fulfilled as soon as $\lambda \leq \frac{\delta_0}{\|\Delta\|}$ which holds asymptotically.

The convergence in Probability directly follows from the above Proposition. ■

When the jump time is deterministic (even with random jumps), we have a more precise result. In this case we have a Central Limit Theorem.

THEOREM 5.1 *Assume that (A0), (A1') and (B1) are satisfied. If* $A \to \infty$ *and* $A\|\Delta\| \to 0$ *then* $\mathbb{P}(B_{A,\Delta}) \to 1$*. Moreover, on the set* $B_{A,\Delta}$ *we have*

$$A^{-1/2}(\rho - c(A, \Delta)) = U_1(A, \Delta) + V_1(A, \Delta) \tag{24}$$

with

$$U_1(A, \Delta) \overset{(\mathcal{L})}{\Rightarrow} \left(\frac{\lambda}{\lambda - 1}\right)^{1/2} \nu_\rho^{-1}\, \mathcal{N}(0, \sqrt{2})$$

where

$$\nu_\rho = \frac{\delta\sigma_\rho^2}{[1/2\sigma^4_{\rho-} + 1/2\sigma^4_{\rho+}]^{1/2}}$$

and for every $\alpha < 1/4$ there exist two constants $K_{\alpha,\lambda}$ and $\tilde{K}_{\alpha,\lambda} > 0$ such that

$$\mathbb{E}V_1(A,\Delta)^2 \leq K_{\alpha,\lambda}\|\Delta\| + \tilde{K}_{\alpha,\lambda}(\|\Delta\|^{\alpha} + A^{-\alpha})$$

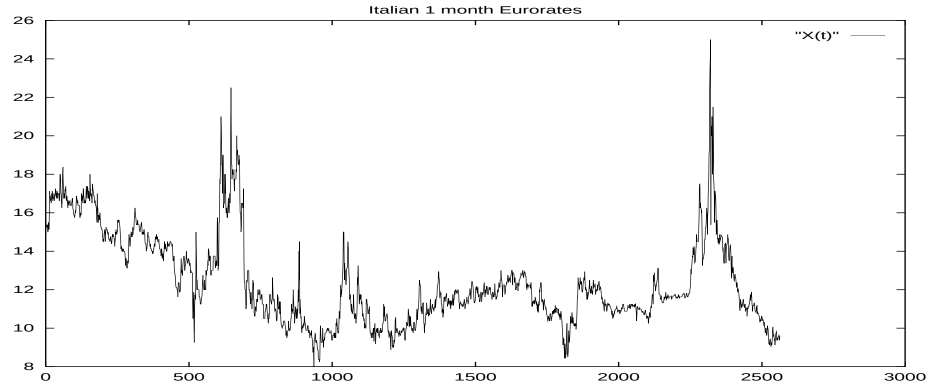
Proof: See Appendix D. ■

Remark: What is the good choice for λ ? Remember that $\forall\lambda \in \mathbb{N}$, $\left(\frac{\lambda}{\lambda-1}\right)^{1/2} > 1$. On the other hand, (A3) may fail for λ too large. Since $\lambda = 3$ gives us $\left(\frac{\lambda}{\lambda-1}\right)^{1/2} = 1.22$, this could be considered as a good choice and it is used in the applications (see next section).

6 What Do We See in Real Data ?

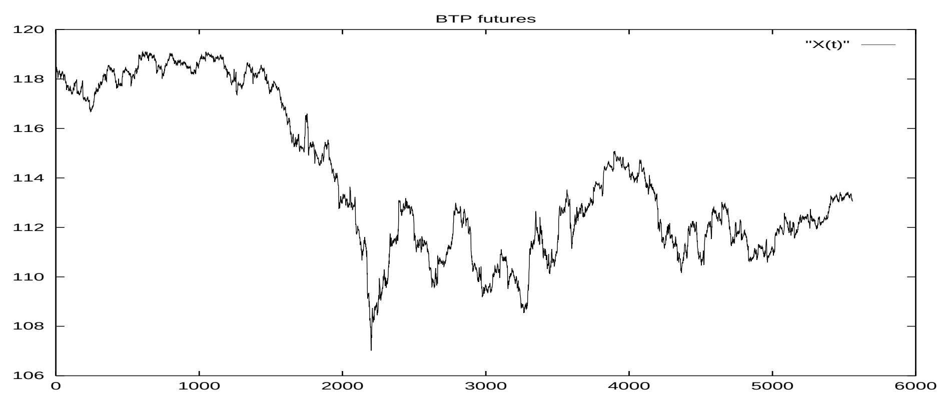
In order to highlight the differencies in data behavior with different time resolutions we use two data sets. The first is the series of the daily observations on one month Italian euro deposit rates from 1984 to 1994. In this period the Italian Euromarket became very liquid and we can observe several shocks. These were caused by the impact on the interest rate of expectations of realignments of the Italian Lira.

Italian 1 month Euro deposit rates 1984-1994



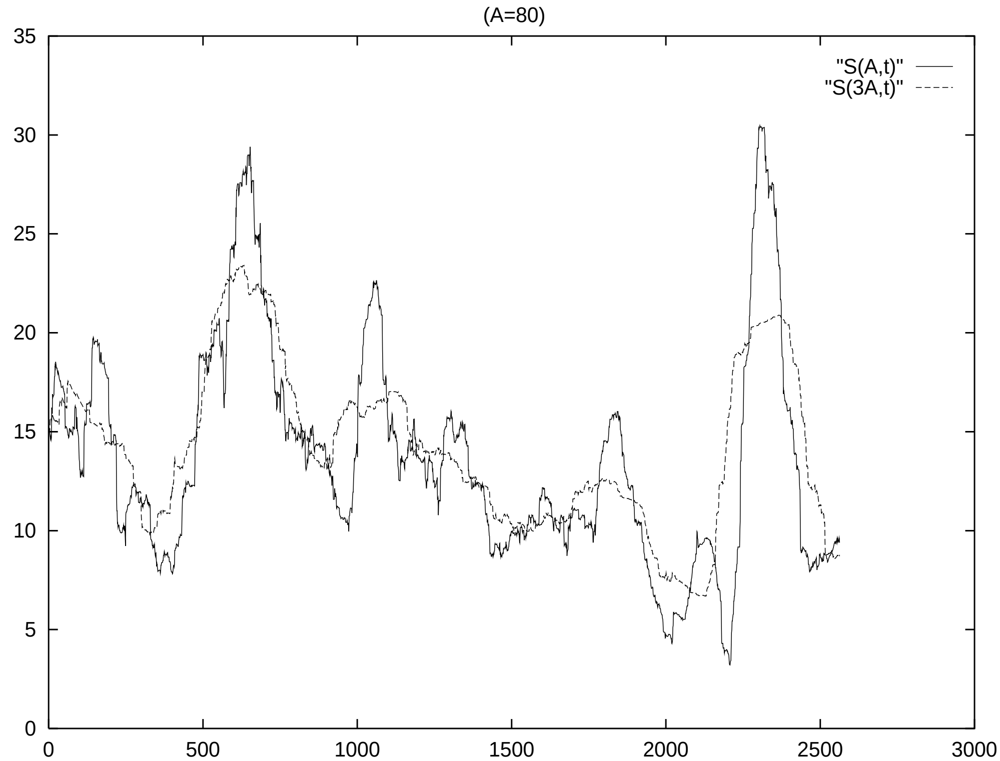
The second series is the Italian BTP (ten year bond) futures price as a five minutes average of quotations from January 1994 to May 1994. For the BTP futures this period was crucial since the political instability which characterized Italy severely hitted in particular the bond market. Secondly, at the begining of 1994 the Federal Reserve suddenly changed the course of its monetary policy and this action had a strong impact on all the major bond markets rising the degree of uncertainty.

Italian BTP futures Jan 1994 May 1994



The observation of these two series seems to suggest a different local behavior of the two data generating processes. In the interest rate series few clear episodes of changes in volatility are visually detectable.

Volatility non-parametric estimations: Italian 1 month Euro deposit rate 1984-1994

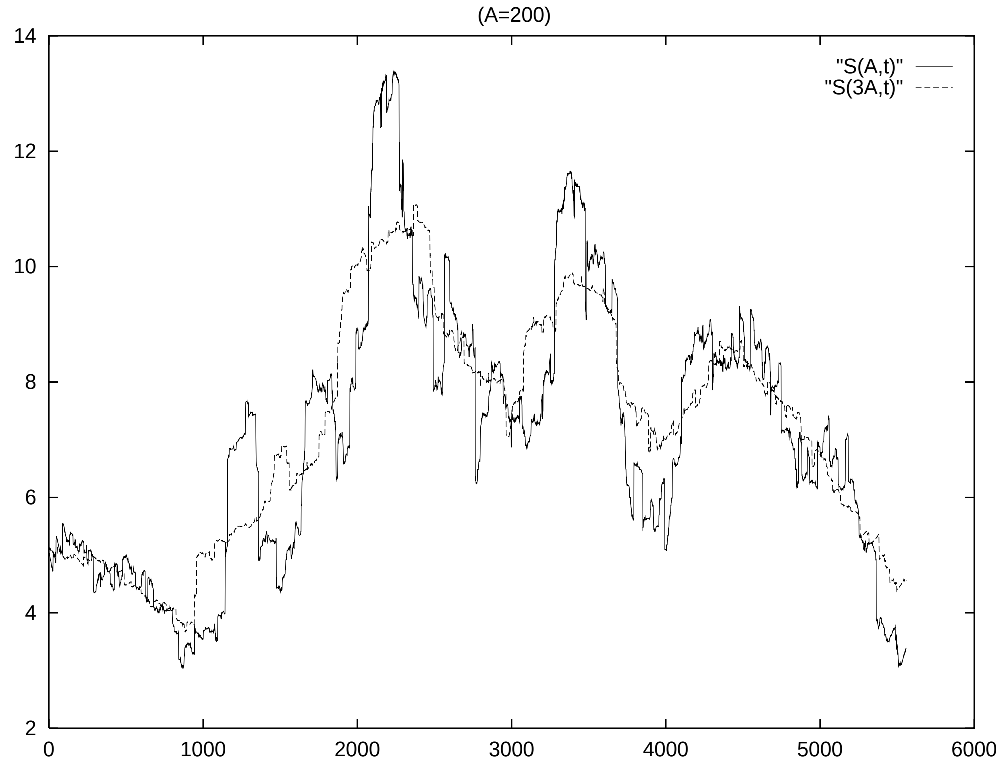


In the BTP futures series instead substantial changes in volatility seems to happen more often.

Volatility non-parametric estimations: Italian BTP futures
Jan 1994-May 1994

The estimations just proposed suggest several interesting considerations. In both cases the proposed estimator seems to track correctly the expected volatility evolutions. For the Eurorates series we observe episodes of volatility explosion the most notable of which is the one corresponding to September 1992. At that time the Italian lira was forced out of the EMS and we had an enormous increase in uncertainty on interest rate markets. For the BTP futures we can see that in addition to the several sudden jumps we have an initial period of increasing volatility followed by a gradual decline.

What seems to be interesting is that for example in daily data (the Eurorates) if we use a window size of 80 days we see clear abrupt changes. At the same time, as should be expected, with a window which is three times bigger (270 days) the volatility evolution becomes smoother and a certain degree of autocorrelation seems to emerge. The same behavior is apparent for the BTP futures. Therefore we are brought back to the initial question: when we talk about persistence which is a reasonable time horizon? Secondly: if the volatility micro structure presents relevant breaks, are we doing the right thing by imposing instead an autoregressive structure?



In any case the crucial question remains that of evaluating the structure of the process followed by the instantaneous volatility.

A APPENDIX (Existence and Positiveness of the Solution of the Cox, Ingersoll and Ross type Stochastic Differential Equation)

The Cox, Ingersoll and Ross model has been used to describe the evolution of financial assets and it is the solution of the following stochastic differential equation:

$$dX_t = c(\alpha - X_t)dt + \sigma(0 \vee X_t)^{1/2}\, dW_t \tag{25}$$

The statistical work done so far in order to identify the values of the coefficients c, α, and σ from real world financial data shows that these parameters are time dependent (Brown and Dybvig [5], Barone et. al. [1], Fournié and Talay [14]). It is therefore natural to consider the Cox, Ingersoll and Ross model with time dependent or stochastic coefficients as in (4).

In this situation we have to ensure that, as in the case of constant coefficients, the solution of this stochastic differential equation exists, is positive and determine under which conditions it does not vanish.

A.1 The Problem and its Solution

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t$ such that (W_t) is an $\mathcal{F}_t$-adapted Wiener process.

Let us recall the result for the case of constant coefficients, see e.g. Ikeda and Watanabe, [24, p.236]):

Proposition A.1 *If c, α and σ are constant then: i) for all initial values $X(0)$, there exists a strong unique solution for equation (25). Moreover, if $c\,\alpha \geq 0$ and $X(0) \geq 0$ almost surely then $\mathbb{P}(X(t) \geq 0, \forall t) = 1$.*
ii) if $\sigma^2 \leq 2c\,\alpha$ and $X(0) > 0$ almost surely, then $\mathbb{P}(e = +\infty) = 1$, where $e(\omega) = inf\{t,\ such\ that\ X(t, \omega) = 0\}$ is the time at which the process X_t reaches the bound 0.

We now generalize Proposition A.1 to the case of time varying coefficients. We need the following assumptions:

(C1) $\theta(t)$, $c(t)$ and $\alpha(t)$ are $(\mathcal{F}_t)$ adapted.

(C2) $\forall N \in \mathbb{N}, \exists K_N, \lambda_N > 0$ such that $\forall t \in [0, N]$,

$$|c(t)|, |\alpha(t)|, |\theta(t)| < K_N \ a.s. \qquad and \qquad |\theta(t)| \geq \lambda_N$$

(C3) $\theta(t)^2 \leq 2\,c(t)\alpha(t)$ $\mathbb{P} \otimes$ Leb -*almost everywhere*

We have the following result.

THEOREM A.1 *If hypothesis (C1), (C2), (C3), are satisfied, $\mathbb{E} \mid X(0) \mid^2 < \infty$ and $X(0) > 0$ a.s. then there exists a unique strong solution of the equation (4). Moreover $\mathbb{P}(X(t) > 0, \forall t) = 1$.*

Proof: The proof is devided into three steps.

First Step : The strong uniqueness of the solution to (4) follows from the Yamada-Watanabe theorem, see for example [25, Prop.2.13, p. 291], after having adapted the proof to our conditions.

Second Step : We assume there exist a **strong** solution on a random interval $[0, \tau[$. This solution turns out to be positive, by an easy adaptation of the proof of Ykeda-Watanabe [24, p. 236]. We will prove the existence of a lower bound, i.e., $\forall t > 0$, $X_t \geq Z_t$ a.s. where the process (Z_t) satisfies a Cox, Ingersoll and Ross stochastic differential equation with constant coefficients and $\forall t > 0$, $Z_t > 0$ a.s.

We want to use the comparison theorem for a process solution of (25), with constant coefficients. We go back to the case of $\sigma(t)$ constant through a time change. We set:

$$M_t = \int_0^t \sigma(s) dW_s, \quad < M >_t = \int_0^t \sigma^2(s) ds$$

Thanks to Time-Change Theorem for martingales see e.g. [25, th 4.6, p.174], we have $M_t = B_{<M>_t}$ with B_s a standard Brownian motion for the filtration $\mathcal{G}_s := \mathcal{F}_{T(s)}$ and $T(s) = inf(t \geq 0, < M >_t > s)$.
Since $\sigma^2(u) > 0$, we have $du = \sigma^{-2}(u) d < M >_u$. Let $Y_s := X_{T(s)}$ we have:

$$\int_0^s (X_v)^{1/2}\, dM_v = \int_0^s (Y_u)^{1/2}\, dB_u$$

From $\theta^2(u) > 0$, we get $< M >_{T(v)} = v$, therefore

$$Y_s = \int_0^{T(s)} \sigma(u)(X_u)^{1/2} dW_u + \int_0^{T(s)} \frac{c(u)}{\sigma^2(u)} \left[\alpha(u) - X_u\right] d < M >_u$$

Following Karatzas and Shreve [25, Prop. 4.8, p.176] if $\int_0^\infty |X(s)| d < M >_s < +\infty$ a.s., we have

$$Y_s = \int_0^s (Y_u)^{1/2}\, dB_u + \int_0^s \frac{c(T(v))}{\sigma^2(T(v))} \left[\alpha(T(v)) - Y_v\right]\, dv$$

This Proposition 4.8 in [25, p. 176] remains true under a local condition $\int_0^N |X(s)| d < M >_s < +\infty$ a.s. for every $N \in \mathbb{N}$, which is verified under our hypothesis, since (X_t) has a.s. continuous trajectories (using (C2)).

Since $Y_s \geq 0$ a.s., Y_s verifies the stochastic differential equation:

$$dY_s = (Y_s)^{1/2} dB_s + \frac{c(T(s))}{\sigma^2(T(s))} \left[\alpha(T(s)) - Y_s\right] ds$$

The Comparison Theorem by Ikeda and Watanabe [24, Th. 1.1, p. 437] can be used even though the drift $b(t, y)$ is not continous in t. In fact we only use the condition that the function $b(t, y)$ be uniformly Lipschitz continuous in y, uniformly in t, almost surely which follows from (C2).

Therefore we have $\mathbb{P}(Y_s \geq Z_s, \forall s > 0) = 1$ with

$$dZ_s = (Z_s)^{1/2} dB_s + \left[\frac{1}{2} - \sup\left(\frac{c(u)}{\sigma^2(u)}\right) Z_s\right] ds$$

and $Z(0) = Y(0) = X(0)$

Since the process Z_t satisfies a Cox, Ingersoll and Ross type stochastic differential equation with constant coefficients, from Proposition A.1, we get $\mathbb{P}(Z_s > 0, \forall s > 0) = 1$ which implies $\mathbb{P}(Y_s \geq 0, \forall s > 0) = 1$.

Third Step : Now we prove the existence of a positive strong solution.

For every $n \in \mathbb{N}$, we define $X_t^{(n)}$ as the solution of the following stochastic differential equation

$$dX_t^{(n)} = c(t)\left[\alpha(t) - X_t^{(n)}\right] dt + \theta(t)\left[\frac{1}{n} \vee X_t^{(n)}\right]^{1/2} dW_t, \qquad X^{(n)}(0) = X_0 \tag{26}$$

We have suppressed the neighbourhood $[0, \frac{1}{n}[$ where the diffusion coefficient vanishes, therefore we get a Lipschitz continuous diffusion coefficient. Since $\mathbb{E}X(0)^2 < +\infty$, there exists a strong solution of the Stochastic Diffrential Equation (26), with almost continuous paths [25, Th.2.9, p.289].

We define the stopping times $\tau_n := inf\left\{t \text{ such that } X_t^{(n)} < \frac{1}{n}\right\}$ and $\tau_\infty := sup_{n\in\mathbb{N}}\tau_n$. >From $X(0) > 0$ a.s. we get $\tau_\infty > 0$ a.s. Therefore, X_t is well defined by $X_t := X_t^{(n)}$ for $t \in [0, \tau_n)$. Of course X_t satisfies the SDE (4) and $X_t \geq 0$ for every $t \in [0, \tau_\infty)$.

It remains to prove that $\tau_\infty = \infty$. This follows from the second step.

Indeed, we have $\mathbb{P}(Y_s \geq Z_s, \forall s > 0 \text{ such that } T(s) < \tau_\infty) = 1$ and $\mathbb{P}(Z_s > 0, \forall s > 0) = 1$. As the solution of the stochastic differential equation (25), Z_s has continuous path, almost surely. Then there exists a sequence of negligible sets S_N such that

$$\forall N \in \mathbb{N},\ \forall\omega \notin S_N,\ \exists m(\omega) > 0 \text{ such that } \forall s > 0,\ Z_s \geq m(\omega)$$

Therefore $\forall N \in \mathbb{N}$, $\forall\omega \notin S_N$, $\exists m(\omega) > 0$ such that $\forall n > m(\omega)^{-1}$, $\forall u \leq inf(N, \tau_n)$, $X_u > \frac{1}{n}$.

But X_u is almost surely continuous on $[0, inf(N, \tau_\infty)[$, thus $\tau_n < N \Rightarrow X_{\tau_n} = \frac{1}{n}$. This induces

$$\forall N \in \mathbb{N},\ \forall\omega \notin S_N,\ \exists m(\omega) > 0 \text{ such that } \forall n > m(\omega)^{-1},\ \tau_n \geq N$$

Outside $S = \bigcup_{N\in\mathbb{N}} S_N$, we have $\tau_\infty = \infty$. Since S is a negligible set, we have proved the existence of a strong solution of the stochastic differential equation (4) on $\mathbb{R}^+$.

This ends the proof and allow us to deduce Theorem 2.1. ∎

B APPENDIX (Some Properties of the Random Variable ξ_i and Bound on η_i)

First we propose a bound on η_i.

Lemma B.1 *Let X_t verify (3) and assume that (A0), (B1) are satisfied. Then $\forall j$ we have*

$$\mathbb{E}(\eta_i^2) \leq 12\sqrt{2}\Delta_i\left[K_T\|\mathbb{E}\sigma^4(\cdot)\|_{L^\infty(0,T)}\right]^{1/2}[1 + \mathcal{O}(\|\Delta\|)]$$

Proof: It follows from Jensen and Hölder inequalities, (14) and the following bound:

$$\mathbb{E}(X_s - X_{t_i})^4 \leq 8(s-t_i)^2 \left[36 \left\|\mathbb{E}\sigma^4(\cdot)\right\|_{L^\infty(t_i,t_{i+1})} + K_T \left\|\Delta\right\|^2\right]$$

■

We now give some useful properties for the family of random variable (ξ_i) defined by (13).

Lemma B.2 *Assume that (A0) is satisfied, then we have:*

$$\mathbb{E}\xi_j = 0, \forall j \qquad \text{and} \qquad \mathbb{E}(\xi_k \xi_l) = 0 \qquad \text{when} \quad k \neq l \tag{27}$$

$$\mathbb{E}\xi_j^2 \leq 3\|\mathbb{E}\sigma^4(\cdot)\|_{L^\infty(t_j,t_{j+1})} \qquad \text{and} \qquad \mathbb{E}\xi_{t_j}^4 \leq C_4\|\mathbb{E}\sigma^8(\cdot)\|_{L^\infty(t_j,t_{j+1})} \tag{28}$$

Proof: Formula (27) follows from (13), assumption (A0) and Stochastic Integral properties. The bounds (28) result from (A0), bound of Stochastic Integral Moment, see e.g. [25, p. 163] and standard calculations using Hölder inequality (see detailed proof in [3]). ■

Using the regularity assumption, we get a better result:

Lemma B.3 *(i) If (A0) and (A2) are satisfied. Then:*

$$\xi_j = \sigma^2(t_j)\Delta_j^{-1} \int_{t_j}^{t_{j+1}} (W_s - W_{t_j})dW_s + \epsilon_j \tag{29}$$

with

$$\mathbb{E}\epsilon_j^2 \leq 3\Delta_j^{2m} \left[\mathbb{E}K^4(\omega)\right]^{1/2} \|\mathbb{E}\sigma^4(\cdot)\|_{L^\infty(0,T)}^{1/2} \left[1 + 12\|\Delta\|\right] \tag{30}$$

Therefore

$$\mathbb{E}\xi_j^2 = \frac{1}{2}\mathbb{E}\sigma^4(t_j) + \mathcal{O}(\|\Delta\|^m) \tag{31}$$

(ii) If (A0) and (A1) are satisfied, then (31) holds with

$$\mathbb{E}\epsilon_j^2 \leq 12\mathbb{P}(t_\rho \in [t_j, t_{j+1}))^{1/2} \|\sigma\|_{L^\infty_{(\Omega\times\mathbb{R})}}^{1/2} \|\ \mathbb{E}\sigma^4(\cdot)\ \|_{L^\infty(0,T)}^{1/2} \left[1 + 12\|\Delta\|\right] \tag{32}$$

and

$$\lim_{h->0} \mathbb{P}(t_\rho \in [t-h, t+h]) = \mathbb{P}(t_\rho = t) \tag{33}$$

Proof: We have:

$$\epsilon_j = \Delta_j^{-1} \int_{t_j}^{t_{j+1}} [\sigma(s) - \sigma(t_j)] \left[\int_{t_j}^{s} \sigma(u)dW_u\right] dW_s$$

Using (13), the bounds of Stochastic Integral moments and Hölder Inequality, a straightforward calculation leads to (30).

The proof of (32) is quite the same. From Lebesgue's Dominated Convergence Theorem, we deduce (33). ■

C APPENDIX (Proof of Pointwise Convergence)

The term $M_{A,\Delta}(t)$ converges to the volatility function $\sigma^2(t)$ exept at the jump times. This is the statement of the following lemma.

Lemma C.1 *i) Assume that (A2) holds, then for every t,*

$$|M_{A,\Delta}(t) - \sigma^2(t)| \leq K(\omega)(A\|\Delta\|)^m$$

ii) Assume that (A1) holds. Then for every t when $A\|\Delta\|$ is small enough

$$M_{A,\Delta}(t) = 1/2\{\sigma^2(t_+) + \sigma^2(t_-)\}$$

Proof: This results from the fact that $M_{A,\Delta}(t)$ is the moving average of size A of $\sigma(t)^2$. ∎

The terms $D_{A,\Delta}(t)$ is of order $\|\Delta\|^{1/2}$, as stated below.

Lemma C.2 *Assume that (A0) and (B1) are satisfied. Then $\forall t > 0$,*

$$\mathbb{E}|D_{A,\Delta}(t)|^2 < 48\sqrt{2}\,\|\Delta\|\left[K_T\|\mathbb{E}\sigma^4(\cdot)\|_{L^\infty(0,T)}\right]^{1/2}[1 + \mathcal{O}(\|\Delta\|)]$$

Proof: From Jensen Inequality we get:

$$\mathbb{E}|D_{A,\Delta}(t_j)|^2 \leq 4A^{-1}\sum_{i=-A/2}^{A/2}\mathbb{E}(\eta_{j+i}^2)$$

The result follows from Lemma B.1. ∎

The term $N_{A,\Delta}(t)$ satisfies a Central Limit Theorem.

Lemma C.3 *(i) Assume that (A0) is satisfied. Then for every t*

$$\mathbb{E} \mid N_{A,\Delta}(t) \mid^2 \leq 12A^{-1}\|E\sigma^4(.)\|_{L^\infty(0,T)} \tag{34}$$

(ii) If, moreover $A \to +\infty$, $A\|\Delta\| \to 0$, and $\exists\nu_0 > 0$ such that $\forall t$, $\sigma(t)^2 \geq \nu_0$, and $\mathbb{P}(\rho = t) = 0$. Then

$$A^{1/2}\,N_{A,\Delta}(t)/\sigma^2(t) \Rightarrow \mathcal{N}(0,\sqrt{2}) \tag{35}$$

Remark : If $\sigma(t)$ is deterministic, we get:

$$A^{1/2}\frac{N_{A,\Delta}(t)}{\sigma^2(t)} \Rightarrow \mathcal{N}(0,\sqrt{2}).$$

This follows from Lindeberg Theorem with Lyapunov condition. Here the difficulty to obtain *(ii)* comes from $\sigma(t)$ stochastic and we need to use the Central Limit Theorem for array martingale.

Proof of Lemma C.3:

(i) Using Lemma A.2 (27) we have $\mathbb{E} \mid N_{A,\Delta}(t_j) \mid^2 = 4A^{-2} \sum_{i=j-A/2}^{j+A/2} \mathbb{E}(\xi_j^2)$. Therefore (34) follows from (28).

(ii) We consider the time t_j as fixed and A, Δ depending on N as defined by the asymptotic. Let

$$S_N := \sigma^{-2}(t_{j-A/2})\, N_{A,\Delta}(t_j) = 2A^{-1} \sum_{i=j-A/2}^{j+A/2} \sigma^{-2}(t_{j-A/2})\, \xi_i$$

and $\mathcal{F}_{N,i} := \mathcal{F}_{t_i}$, the random variables ξ_i are $\mathcal{F}_{N,i+1}$ adapted. Since $\sigma^2(t_{j-A/2})$ is $\mathcal{F}_{N,i}$ adapted (for every $i \geq j - A/2$) and $\mathbb{E}(\xi_i \mid \mathcal{F}_{N,i}) = 0$, S_N is a martingale array. Anyway the filtrations are not nested i.e. Condition (3.21) [19, p.58] is not satisfied. To avoid this difficulty, we prove that the Conditional Variance V_N^2 has, in Probability, a deterministic limit [19, p.59]. Indeed, we have:

$$V_N^2 := 4A^{-2} \sum_{i=j-A/2}^{j+A/2} \mathbb{E}(\sigma^{-4}(t_{j-A/2})\, \xi_i^2 \mid \mathcal{F}_{N,i}) =$$

$$= 4A^{-2}\, \sigma^{-4}(t_{j-A/2}) \sum_{i=j-A/2}^{j+A/2} \mathbb{E}(\xi_i^2 \mid \mathcal{F}_{N,i})$$

From Lemma B.3 and (29), we get:

$$\mathbb{E}(\xi_i^2 \mid \mathcal{F}_{N,i}) = \frac{1}{2}\sigma^4(t_{j-A/2}) + \lambda_i$$

Therefore

$$V_n^2 = 2A^{-1}\,[1 + Res]$$

with

$$Res = 4A^{-1} \sum_{i=j-A/2}^{j+A/2} \sigma^{-4}(t_{j-A/2})\lambda_i$$

where exact formula for λ_i could be derived from Lemma B.3.

If (A2) holds, we have $\mathbb{E}(Res^2) = \mathcal{O}(A\|\Delta\|)^{2m} \to 0$ as $A\|\Delta\| \to 0$. If (A1) holds, we have $\mathbb{E}(Res^2) = \mathcal{O}(\mathbb{P}(\rho \in [t_{j-A/2}, t_{j+A/2}])^{1/4}) \to 0$ from (32). In both case, we apply Central Limit Theorem for array martingale [19, cor.3.1, p.58] (after normalization by $A^{1/2}$) and we get (35). The verification of Lyapunov Condition is equivalent to $\lim_{N\to\infty} A^{-1} = 0$ and follows from (28).

∎

D APPENDIX (Rate of Convergence of the Volatility Jump Time Estimator)

With a little abuse of notation, we still denote $g_A(\cdot)$, $L_A(\cdot)$ and $\mathcal{D}_A(\cdot)$ the functions defined by (21) but with argument t_j instead j. Recall that we are considering the asymptotic $A \to \infty$, therefore when (A1') is fulfilled (A3) holds for $A\|\Delta\|$ small enough (see proof of Corollary 5.2). When (A3) is fulfilled, on the set $B_{A,\Delta}$ we have

$$\left(\frac{\rho - c}{A}\right)(1-\lambda^{-1})(\delta\sigma_\rho^2) = A^{-1/2} L_A(t_c) - \mathcal{D}_A(t_c) \tag{36}$$

This follows from (19) and (21). For each fixed time t_j, we have

$$L_A(t_j) \stackrel{\mathcal{L}}{\Rightarrow} \left(\frac{\lambda-1}{\lambda}\right)^{1/2} [\sigma_{\rho-}^4 + \sigma_{\rho+}^4]^{1/2} \mathcal{N}(0,1)$$

when $A \to +\infty$. Since $t_c(A,\Delta)$ is a random variable, we cannot deduce directly the Theorem 5.1. Therefore we proceed as follows (using the same kind of strategy as in [4]).

1) we show that $t_c(A,\Delta)$ converges to t_ρ (in $L^2(\Omega)$ norm);

2) we show asymptotic normality with $t_c(A,\Delta)$ replaced by t_ρ;

3) we bound the error by showing a Hölder continuity result with the help of Kolmogorov's Lemma.

Recall that the jump times t_ρ are deterministic (by assumption of Theorem 5.1). For notational convenience, we will denote t_c instead $t_c(A,\Delta)$. We first state the Hölder continuity and boundness results, in the following lemma.

Lemma D.1 *Assume that (A0), (A1') and (B1) are satisfied. Then we have*

i) There exists $K_1 > 0$ *such that* $\mathbb{E}(\mathbf{1}_{\{|c-\rho|\le A/2\}}\mathcal{D}_A^2(t_c)) \le K_1\,\|\Delta\|$;

ii) $\forall\alpha < 1/4, \exists K_2 > 0$ *such that* $\mathbb{E}M_\omega^4 \le K_2$ *and a negligeable set* N_A *such that* $\forall\omega \notin N_A$, $\forall j,k \in [0,N]\cap\mathbb{N}$,

$$|L_A(t_j) - L_A(t_k)| \le M_\omega\, |\frac{j-k}{A}|^\alpha$$

iii) there $\exists K_3 > 0$ *such that* $\mathbb{E}(\mathbf{1}_{\{|c-\rho|\le A/2\}}\, L(t_c)^2) \le K_3$.

Proof :

(i) We want to apply the Kolmogorov Lemma in order to obtain the continuity of $\mathcal{D}_A(t_j)$. If we can show that

$$\forall j,k \in [0,N]\cap\mathbb{N}, \qquad \mathbb{E}|\mathcal{D}_A(t_j) - \mathcal{D}_A(t_k)|^2 \le \|\Delta\| K_0 \left(\frac{k-j}{A}\right)^2 \tag{37}$$

then for every $\alpha < 1/2$, there exists a random variable M_ω and a constant $\gamma(\alpha) > 0$ such that $\mathbb{E}M_\omega^2 \le \gamma(\alpha)\,K_0\,\|\Delta\|$ and

$$\forall j,k \in [0,N]\cap\mathbb{N}, \qquad |\mathcal{D}_A(t_j) - \mathcal{D}_A(t_k)| \le M_\omega \left|\frac{k-j}{A}\right|^\alpha \qquad a.s.$$

Combined with Lemma C.2, this induces

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{|\rho-c|\le A/2\}}\mathcal{D}_A^2(t_c)\right] &\le 2\mathbb{E}\left(\mathbf{1}_{\{|\rho-c|\le A/2\}}|\mathcal{D}_A(t_c)-\mathcal{D}_A(\rho)|^2\right)+2\mathbb{E}\mathcal{D}_A^2(\rho)\\
&\le \mathbb{E}M_\omega^2+2\mathbb{E}\mathcal{D}_A^2(\rho)\\
&\le K_1\|\Delta\|
\end{aligned}
$$

In order to show (37), we get 4 terms of the type

$$
\mathbb{E}\left|A^{-1}\sum_{i=j-A/2}^{k-A/2}\eta_i\right|^2 = A^{-2}\sum_{i_1,i_2}\mathbb{E}(\eta_{i_1}\eta_{i_2})
$$

$$
\le A^{-2}(k-j)^2\sup_i(\mathbb{E}\eta_i^2)
$$

After we deduce (37) by using Lemma B.1

(ii) It directely follows from Kolmogorov's Lemma. So it suffices to verify

$$
\forall j,k\in[0,N]\cap\mathbb{N},\qquad \mathbb{E}|L_A(k)-L_A(t_j)|^4\le\gamma\left(\frac{k-j}{A}\right)^2
$$

Here we have 4 terms of the type

$$
\begin{aligned}
A^{-2}\mathbb{E}\left|\sum_{i=j-A/2}^{k-A/2}\xi_i\right|^4 &= A^{-2}\sum_{i_1,i_2}\mathbb{E}(\xi_{i_1}^2\xi_{i_2}^2)\\
&\le \|\sigma\|_{L^\infty(0,T)}^4 CA^{-2}(j-k)^2
\end{aligned}
$$

where we have used the independency of the ξ_i and Lemma B.2.

(iii) We proceed in the same way as in (i). ■

Remark : We are not using the Kolmogorov's Lemma stated in Revuz & Yor [32, th. 2.1, p.25]. Here we need a bound for a moment of the Hölder continuity constant (for e.g. $\mathbb{E}M_\omega^2\le\gamma(\alpha)\,K\,\|\Delta\|$ in (i)). This improvement could be easily deduced from a careful reading of the proof of Revuz & Yor. Moreover, since for each fixed A and Δ there is only a finite number of times t_i, we do not need all the power of Kolmogorov's Lemma. But it is more convenient to use it.

Now we can yield the first step:

Lemma D.2 *Assume that (A0), (A1'), (B1) and (A3) are satisfied. Then there exist* $K_3>0$ *such that:*

$$
\mathbb{E}\left(\mathbf{1}_{B_{A,\Delta}}\left|\frac{\rho-c}{A}\right|\right)\le K_3\,(\delta\sigma_\rho^2)^{-1}\,\left\{\|\Delta\|^{1/2}+A^{-1/2}\right\}
$$

Proof : From (36), Jensen Inequality and Lemma D.1, we get:

$$\begin{aligned}
&(\delta\sigma^2_\rho)\,(1-\lambda^{-1})\,\mathbb{E}\left(\mathbf{1}_{B_{A,\Delta}}\left|\tfrac{\rho-c}{A}\right|\right)\\
&\qquad \le\; A^{-1/2}\mathbb{E}\left(\mathbf{1}_{\{|\rho-c|\le A/2\}}\left|\mathcal{L}_{A,\Delta}(t_c)\right|\right)+\mathbb{E}\left(\mathbf{1}_{\{|\rho-c|\le A/2\}}\left|\mathcal{D}_{A,\Delta}(t_c)\right|\right)\\
&\qquad \le\; A^{-1/2}\mathbb{E}\left(\mathbf{1}_{\{|\rho-c|\le A/2\}}\mathcal{L}^2_{A,\Delta}(t_c)\right)^{1/2}+\mathbb{E}\left(\mathbf{1}_{\{|\rho-c|\le A/2\}}\mathcal{D}^2_{A,\Delta}(t_c)\right)^{1/2}\\
&\qquad \le\; K_3\left\{\|\Delta\|^{1/2}+A^{-1/2}\right\}.
\end{aligned}$$

■

We have the second step:

Lemma D.3 *Assume that (A0), (A1'), (B1) and (A3) are satisfied. If $A\to\infty$, then*

$$L_A(t_\rho)\;\overset{\mathcal{L}}{\Rightarrow}\;\left(\frac{\lambda-1}{\lambda}\right)^{1/2}[\sigma^4_{\rho-}+\sigma^4_{\rho+}]^{1/2}\mathcal{N}(0,1)$$

Proof : From (21) and (10), we have

$$L_A(t_j)=2A^{-1/2}\lambda^{-1}\left\{(\lambda-1)\sum_{i=-A/2}^{A/2}\xi_{j+i}-\sum_{\substack{i=-\lambda A/2\\|i|>A/2}}^{\lambda A/2}\xi_{j+i}\right\}$$

Recall that in the deterministic case, the random variables ξ_j are independent. We obtain:

$$\begin{aligned}
S_A &:= 4A^{-1}\lambda^{-2}\left\{(\lambda-1)^2\sum_{i=-A/2}^{A/2}\mathbb{E}\xi^2_{j+i}+\sum_{\substack{i=-\lambda A/2\\|i|>A/2}}^{\lambda A/2}\mathbb{E}\xi^2_{j+i}\right\}\\
&= 4A^{-1}\lambda^{-2}\left\{(\lambda-1)^2\frac{A}{2}\tilde\sigma^4_\rho+(\lambda-1)\frac{A}{2}\tilde\sigma^4_\rho\right\}
\end{aligned}$$

where

$$\tilde\sigma^4_\rho=\frac{1}{2}[\sigma^4_{\rho-}+\sigma^4_{\rho+}]$$

Finally

$$S_A=2\lambda^{-1}(\lambda-1)\tilde\sigma^4_\rho$$

The Lyapunov condition is satisfied if $A^{-1}\to 0$. Therefore we can deduce from the Central Limit Theorem:

$$L_A(\rho)\Rightarrow\mathcal{N}\left(0,\left(2\,\frac{\lambda-1}{\lambda}\,\tilde\sigma^4_\rho\right)^{1/2}\right)$$

and the lemma is proved. ■

Finally we have the third step:

Proof of Theorem 5.1 :

The assumption (A3) is satisfied when $A\|\Delta\| \to 0$, see the proof of Corollary 5.2. From (36), we get

$$A^{-1/2}(\rho - c)\,(1-\lambda^{-1})\,(\delta\sigma_\rho^2) = L_A(\rho) + A^{-1/2}\mathcal{D}_A(t_c) + \{L_A(t_c) - L_A(t_\rho)\}$$

Therefore we have (24) where

$$U_1(A) := \frac{\lambda}{\lambda - 1}(\delta\sigma_\rho^2)^{-1}L_A(t_\rho)$$

and

$$V_1(A) := \left(\frac{\lambda}{\lambda-1}\right)(\delta\sigma_\rho^2)^{-1}\left\{A^{-1/2}\mathcal{D}_A(t_c) + \{L_A(t_c) - L_A(t_\rho)\}\right\}$$

The convergence in law of U_1 is given by Lemma D.3

From Lemma D.1 (i), we have

$$I\!\!E\left[\mathbf{1}_{\{|\rho - c| \le A/2\}}A^{1/2}\mathcal{D}_A(t_c)\right]^2 \le K_1\,A\|\Delta\|$$

On the other hand, using successively Lemma D.1 (ii), Hölder Inequality, Jensen Inequality and Lemma D.2, we get:

$$\begin{aligned}
I\!\!E\left(\mathbf{1}_{B_{A,\Delta}}[L_A(t_c) - L_A(t_\rho)]\right)^2 &\le I\!\!E\left(\mathbf{1}_{B_{A,\Delta}}M_\omega^2\left|\frac{c-\rho}{A}\right|^{2\alpha}\right) \\
&\le (I\!\!E M_\omega^4)^{1/2}\,I\!\!E\left(\mathbf{1}_{B_{A,\Delta}}\left|\frac{c-\rho}{A}\right|^{4\alpha}\right)^{1/2} \\
&\le K_2^{1/2}\,I\!\!E\left(\mathbf{1}_{B_{A,\Delta}}\left|\frac{c-\rho}{A}\right|\right)^{2\alpha} \\
&\le K_4\left\{\|\Delta\|^{1/2} + A^{-1/2}\right\}^{2\alpha}
\end{aligned}$$

Let us stress that the last inequality does not directly follow from Lemma D.1 (i), but we need the improved bound of Lemma D.2.

This ends the proof of Theorem 5.1. ■

References

[1] E. BARONE, D. CUOCO and E. ZAUTZIK, “La struttura dei rendimenti per scadenza secondo il modello di Cox, Ingersoll e Ross: una verifica empirica”, *Temi di discussione* **128**, Banca d'Italia, (1989).

[2] A.BENSOUSSAN, M.CROUHY and D.GALAY, ”Stochastic equity volatility related to the leverage effect I: Equity volatility behavior,” *Applied Mathematical Finance* **1**, p.63-85 (1994).

[3] P. BERTRAND, “Estimation of the Stochatic Volatility of a Diffusion Process I. Comparison of Haar basis Estimator and Kernel Estimators.” Rapport Technique INRIA No 2739, (1996).

[4] P. BERTRAND and Yu. A. KUTOYANTS, “A Minimum Distance Estimator for Partially Observed Linear Stochastic Systems” to appear in *Statistics & Decision* **14**, (1996).

[5] S.J. BROWN and P.H. DYBWIG, “The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates,” *Journal of Finance* **41**, 3, (1986)

[6] P.BRUGIÈRE, “Théorème de Limite Central pour un estimateur non paramétrique de la variance d'un processus de diffusion multidimensionelle”, *Annales de l'I.H.P. Probabilité & Statistiques*, vol. **29**, 3, p.357-389 (1993).

[7] L.COUTIN, "Estimation des coefficients d'un processus càdlàg observé à des temps discrets", Pub. I.S.U.P. **38** fasc.3, p.87-109, (1994).

[8] J. COX, J. INGERSOLL and S. ROSS, “A Theory of the Term Structure of Interest Rates.”, *Econometrica* **53**, p.363-84 (1985).

[9] N. EL KAROUI and M. JEANBLANC-PICQUÉ, “Robustness of the Black and Scholes formula”, preprint **3**, Université d'Evry (1994).

[10] D.DACUNHA-CASTELLE and D.FLORENS-ZMIROU, “Estimation of the coefficient of diffusion from discrete observations”, *Stochastic* **19**, p.263-284, (1986).

[11] G.DOHNAL, “On Estimationg the Diffusion coefficient”, *Journal of Applied Probability* **24**, p.105-114, (1987).

[12] D.FLORENS-ZMIROU, “Estimation de la variance d'une diffusion à partir d'une observation discrétisée”, C.R.A.S., t.309, Série I, p.195-200, (1989).

[13] D.FLORENS-ZMIROU, “On estimating the Variance of diffusion Processes”, *Journal of Applied Probability* **30**, p.790-804, (1993).

[14] E.FOURNIÉ and D.TALAY: "Application de la statistique des diffusions à un modèle de taux d'intérêt", *Finance* Vol. 12, **2** (1991) p.79-111.

[15] V.GENON-CATALOT, C.LAREDO and D.PICARD, "Non-parametric Estimation of the Diffusion coefficient by Wavelets Methods", *Scandinavian Journal of Statistic* **19**, p.317-335, (1992).

[16] V.GENON-CATALOT and J. JACOD, "On the estimation of the Diffusion coefficient for multi-dimensional Diffusion Processes", *Annales de l'I.H.P. Probabilité & Statistiques*, p.119-151, (1993).

[17] C. A. GOODHART, T. ITO and R. PAYNE,"One Day in June, 1993: a Study of the Working of the Reuters Dealing 2000-2 Electronic Foreign Exchange Trading System", *NBER Technical Working Paper* (1994).

[18] C. GOURIEROUX, A. MONFORT and E. RENAULT, "Indirect Inference", *Journal of Applied Econometrics* **8**, p.85-118, (1993).

[19] P. HALL and C.C. HEYDE, *Martingal Limit Theory and its applications*, Academic Press, San Diego (1980).

[20] J. HAMILTON and R. SUSMEL,"Autoregressive Conditional Heteroskedasticity and Changes in Regime", *Journal of Econometrics* **64**, (1994).

[21] A. HARVEY, E. RUIZ and N. SHEPHARD,"Multivariate Stochastic Variance Models", *Review of Economic Studies* **61**, p.247-264, (1994).

[22] J.HULL and A. WHITE "Pricing Interest Rate Derivative Securities," *Review of Financial Studies* **3**, p.573-92, (1990).

[23] I.A. IBRAGIMOV and R.Z. HAS'MINSKII, *Statistical Estimation, Asymptotic Theory*, Springer Verlag, New-York (1981).

[24] N. IKEDA and S. WATANABE, *Equation and Diffusion Processes*, 2^{nd} Edition, North Holland, Amsterdam (1989).

[25] I. KARATZAS and S.E. SHREVE, *Brownian Motion and Stochastic Calculus*, Springer Verlag, New-York (1988).

[26] D. LAMBERTON and B.LAPEYRE, *Introduction au calcul stochastique appliqué à la finance*, Ellipse, Paris, (1991).

[27] L. LE CAM and G.L. YANG, *Asymptotics in Statistics*, Springer Verlag, New-York (1990).

[28] R.S. LIPSTER and A. N. SHIRYAEV, *Statistics of Random Processes*, Springer-Verlag, New York, (1977).

[29] B.B. MANDELBROT, "The Variation of Certain Speculative Prices", *Journal of Business*, **36**, p.394-419, 1963.

[30] U.A. MÜLLER, M. DACOROGNA, R. D. DAVÉ, R. B. OLSEN, O. V.PICTET, J. E. WEIZSÄCKER, "Volatilities of Different Time Resolutions. Analysing the Dynamics of Market Components", Olsen & Associate Research Group, Paper presented at the First International Conference On High Frequency Data in Finance, (1995).

[31] D.B. NELSON, "ARCH Models as Diffusion Approximation" *Journal of Econometrics* **45**, 7-39, (1990).

[32] A. REVUZ and M.YOR, *Continuous Martingales and Brownian Motion*, Springer Verlag, New-York (1991).

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